MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2023-2024

Séance 3 : Représentations du groupe dihédral et d'un produit tensoriel, sous-espace invariants

1. Reality of characters

Since the character only depends on the conjugacy class (it is a class function), we know that $\chi_i(g) = \chi_i(g^{-1})$ if g and g^{-1} are in the same conjugacy class. Without loss of generality, we can assume the representation is unitary (always possible for a finite representation of a finite or compact group), then:

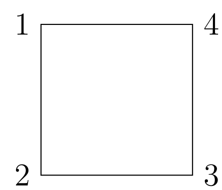
$$\chi_i(g^{-1}) = \mathrm{Tr}\, T_i(g^{-1}) = \mathrm{Tr}\,(T_i(g))^{-1} = \mathrm{Tr}\,(T_i(g))^{\dagger} = (\mathrm{Tr}\, T_i(g))^{\star} = \chi_i(g)^{\star} \ .$$

Using $\chi_i(g) = \chi_i(g^{-1})$ for a self inverse conjugacy class, we conclude $\chi_i(g) \in \mathbb{R}$. The previous result also shows that, in the general case where conjugacy classes are not self inverse, the character of an element and of its inverse are always complex conjugate.

2. D_4 irreducible representations

Section a)

Picture a square as follows:



We have a $\mathbb{Z}_4$ subgroup generated by an element a, which we think of as a $\pi/2$ rotation in a counterclockwise direction (so $a = (1\,2\,3\,4)$ understood as a permutation). Then we have four possible reflections: t_x (reflection along the horizontal axis), t_y (reflection along the vertical axis), t_{13} (reflection along the $1 \leftrightarrow 3$ line), and t_{24} (reflection along the $2 \leftrightarrow 4$ line). Convince yourself that:

$$t_x a = t_{13} \ , \qquad t_x a^2 = t_{13} a = t_y \ , \qquad t_x a^3 = t_y a = t_{24} \ .$$

Furthermore, any reflection satisfies $t^2 = e$, and the commutation of t_x with the rotation is $t_x a = t_{13} = a^3 t_x$. This is enough information to fill in the product table. We will call $b = t_x$ the reflection along the horizontal axis:

	a	a^2	a^3	b	ba	ba^2	ba^3
a	a^2	a^3	e	ba^3	b	ba	ba^2
a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	e	a	a^2	ba	ba^2	ba^3	b
$t_x = b$	ba	ba^2	ba^3	e	a	a^2	a^3
$t_{13} = ba$	ba^2	ba^3	b	a^3	e	a	a^2
$t_y = ba^2$	ba^3	b	ba^2	a^2	a^3	e	a
$t_{24} = ba^3$	b	ba	ba^2	a	a^2	a^3	e

Section b)

There is a hint in the question which says that two conjugate elements have the same order. This is easy to understand if we note that for $g' = hgh^{-1}$ we have $(g')^n = hg^nh^{-1}$ (so $g^n = e$ if and only if $(g')^n = e$). Let us classify the nontrivial elements of D_4 depending on their order:

- Order 4: a, a^3.
- Order 2: a^2, b, ba, ba^2, ba^3.

There is another thing we can see from the previous product table: $a^2b = ba^2$, and as a consequence a^2 commutes with all the elements of the group (since it commutes with the elements of the $\mathbb{Z}_4$ subgroup trivially). Thus, the center is $Z(D_4) = \{e, a^2\}$, and these two elements will form single-element conjugacy classes (the center is an Abelian subgroup). For the remaining elements we have to study case by case. Clearly $bab^{-1} = bab = a^3$, so the two order 4 elements (a and a^3) form a conjugacy class together. The most tedious ones are the remaining order 2 reflections. Try conjugating them by all possible elements (*e.g.*, gbg^{-1} for all $g \in D_4$), and convince yourself that there are 5 conjugacy classes:

$$[e] = \{e\} \ , \quad [a^2] = \{a^2\} \ , \quad [a] = \{a, a^3\} \ , \quad [b] = \{b, ba^2\} \ , \quad [ba] = \{ba, ba^3\} \ .$$

By Burnside's theorem, we will have 5 inequivalent irreps and they will satisfy:

$$8 = 1 + n_1^2 + n_2^2 + n_3^2 + n_4^2 \ .$$

The only possible solution is $n_1 = n_2 = n_3 = 1$, and $n_4 = 2$. To fill in the character table, we will need some information. Summarizing the main steps:

- We always have the trivial representation, which we call T^0.
- As said in the question, we also have the sign representation, inherited from viewing the D_4 elements as permutations of S_4. For example, $a = (1\,2\,3\,4) = (1\,4)(1\,3)(1\,2)$ is represented by -1 (odd number of transpositions), while $b = (1\,2)(3\,4)$ is represented by $+1$. This is T^1 in the table below.

- More generally, notice that in any 1-dimensional representation (denoting $\lambda_g \in \mathbb{C}$ the representatives to emphasize they are just numbers) we have $\lambda_a = \lambda_{a^3}$ because they are in the same conjugacy class. But then $\lambda_a = \lambda_a \lambda_{a^2}$, which forces to have $\lambda_{a^2} = 1$. We emphasize this is true in one-dimensional representations, where representatives only depend on the conjugacy class. There are five conjugacy classes, so five different representatives, and we know $\lambda_e = \lambda_{a^2} = 1$ always. We can only choose freely λ_a and λ_b, because then necessarily $\lambda_{ba} = \lambda_b \lambda_a$. Furthermore, $\lambda_a^2 = \lambda_{a^2} = 1 = \lambda_b^2$, so $\lambda_a = \pm 1$ and $\lambda_b = \pm 1$. There are four different ways to choose these values, and those correspond to the 4 different 1-dimensional representations.

- The characters of the 2-dimensional representation are then obtained by orthogonality.

If you carefully follow the previous reasoning, you should obtain the following character table:

	$[e]$	$[a^2]$	$[a]$	$[b]$	$[ba]$
χ_0	1	1	1	1	1
χ_1	1	1	-1	1	-1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	-1	1
χ_4	2	-2	0	0	0

As you can guess, the 2-dimensional representation is just given by the matrices which implement the geometric transformations of the square (rotations and reflections).

3. Tensor product representations

Preliminaries

Just to set the stage, let us review what is a tensor product space. Given two vector spaces V_1 and V_2 with bases $B_1 = \{e_i \;/\; i = 1, \dots, d_1\}$ and $B_2 = \{f_a \;/\; a = 1, \dots, d_2\}$, the tensor product $V_1 \otimes V_2$ is the vector space generated as:

$$V_1 \otimes V_2 = \mathrm{span}\{e_i \otimes f_a \;/\; i = 1, \dots, d_1;\; a = 1, \dots, d_2\} \;.$$

It is clear from this definition that the dimension of $V_1 \otimes V_2$ is the product of dimensions $d_1 d_2$. We have a bilinear map from $V_1 \times V_2$ to $V_1 \otimes V_2$, since for any two vectors $v \in V_1$ and $w \in V_2$:

$$v \otimes w = \left(\sum_i v^i e_i\right) \otimes \left(\sum_a w^a f_a\right) = \sum_{i,a} v^i w^a \left(e_i \otimes f_a\right) \;.$$

Notice however that not all elements of $V_1 \otimes V_2$ are of the form $v \otimes w$, we allow arbitrary linear combinations of the basis elements $\sum_{i,a} M^{ia} \left(e_i \otimes f_a\right)$ (said it another way, in the product $v \otimes w$ there are only $d_1 + d_2$ degrees of freedom, the components of each of the vectors, while an element of $V_1 \otimes V_2$ has $d_1 d_2$ independent components). Given linear maps $F_1 : V_1 \to V_1'$ and $F_2 : V_2 \to V_2'$, we define the tensor product map $F_1 \otimes F_2 : V_1 \otimes V_2 \to V_1' \otimes V_2'$ by demanding that:

$$(F_1 \otimes F_2)(v \otimes w) = F_1(v) \otimes F_2(w) \ ,$$

for any $v \in V_1$, $w \in V_2$ (this is true in particular of basis elements), and then extending by linearity (there is a single linear map satisfying the previous condition). Notice in particular that the tensor product of linear maps is linear as a map in $V_1 \otimes V_2$, which means it is also linear in each of the factors individually:

$$(F_1 \otimes F_2)\left(\sum_{i,a} M^{ia} e_i \otimes f_a\right) = \sum_{i,a} M^{ia} (F_1 \otimes F_2)(e_i \otimes f_a) = \sum_{i,a} M^{ia} (F_1(e_i) \otimes F_2(f_a)) \ ,$$

$$(F_1 \otimes F_2)((\lambda u + \mu v) \otimes w) = F_1(\lambda u + \mu v) \otimes F_2(w) = \lambda F_1(u) \otimes F_2(w) + \mu F_1(v) \otimes F_2(w) \ .$$

Convince yourself that your are comfortable with these manipulations, in the end it is all very natural if you think about $\otimes$ as an actual product. As a final comment, the matrix components of $F_1 \times F_2$ in a given basis are what you would usually call the tensor product of matrices. Let $F_1(e_i) = \sum_j P_{ji} e_j'$ and $F_2(f_a) = \sum_b Q_{ba} f_b'$, then the matrix components of $F_1 \otimes F_2$ in the bases $e_i \otimes f_a$ and $e_j' \otimes f_b'$ are:

$$(F_1 \otimes F_2)_{jb,ia} = P_{ji} Q_{ba} \ .$$

Section a)

The previous discussion was only to review a bit the notation and the concepts. Consider now, given two representations T^1 and T^2 of a group G with representation spaces V_1 and V_2, the map $T^1 \otimes T^2 : G \to GL(V_1 \otimes V_2)$ which assigns to each $g \in G$ the linear map $T^1(g) \otimes T^2(g)$ from $V_1 \otimes V_2$ into itself. We have to check that it is a valid representation of G, so we verify the two conditions:

- $\{T^1(g) \otimes T^2(g) \ / \ g \in G\}$ are one-to-one linear maps in $V_1 \otimes V_2$. They are linear by definition of the tensor product of linear maps, and for any $T^1(g) \otimes T^2(g)$ it is easy to check that $T^1(g^{-1}) \otimes T^2(g^{-1})$ is the inverse, so they are one-to-one.

- The representatives respect the group multiplication law. This is shown as follows. For any $g, h \in G$, we compute the product of representatives and check it equals the representative of the product:

$$\begin{aligned}\left(T^1(g) \otimes T^2(g)\right)\left(T^1(h) \otimes T^2(h)\right) &= \left(T^1(g) T^1(h)\right) \otimes \left(T^2(g) T^2(h)\right) \\ &= T^1(gh) \otimes T^2(gh) \\ &= \left(T^1 \otimes T^2\right)(gh) \ .\end{aligned}$$

Section b)

We now compute the characters of the tensor product representation. The key step will be to understand what is the trace of the tensor product of two linear maps, so let us do it in general. As before, let $F_1 : V_1 \to V_1$ and $F_2 : V_2 \to V_2$ be two linear maps, and take bases $B_1 = \{e_i \ / \ i = 1, \dots, d_1\}$ and $B_2 = \{f_a \ / \ a = 1, \dots, d_2\}$ for V_1 and V_2. Since $(F_1 \otimes F_2)(e_i \otimes f_a) = \sum_{j,b} (F_1)_{ji}(F_2)_{ba} e_j \otimes f_b$, by definition of the trace:

$$\mathrm{Tr}_{V_1 \otimes V_2} (F_1 \otimes F_2) = \sum_{i,a} (F_1)_{ii}(F_2)_{aa} = \mathrm{Tr}_{V_1}(F_1)\mathrm{Tr}_{V_2}(F_2) \ .$$

The result for the characters is now immediate. For any $g \in G$:

$$\chi_{T^1 \otimes T^2}(g) = \mathrm{Tr}_{V_1 \otimes V_2} \left(T^1(g) \otimes T^2(g)\right) = \mathrm{Tr}_{V_1} \left(T^1(g)\right) \mathrm{Tr}_{V_2} \left(T^2(g)\right) = \chi_{T^1}(g)\chi_{T^2}(g) \ .$$

4. Tensor product representations of S_3

Section a)

The table of characters for tensor product representations can be obtained immediately from the irreps:

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$
χ_{T^0}	1	1	1
χ_{T^1}	1	-1	1
χ_{T^2}	2	0	-1

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$
$\chi_{T^0 \otimes T^0}$	1	1	1
$\chi_{T^0 \otimes T^1}$	1	-1	1
$\chi_{T^0 \otimes T^2}$	2	0	-1
$\chi_{T^1 \otimes T^1}$	1	1	1
$\chi_{T^1 \otimes T^2}$	2	0	-1
$\chi_{T^2 \otimes T^2}$	4	0	1

Notice that:

- Products with the trivial representation give that same representation, *i.e.*, $T^0 \otimes T^i = T^i$. This is just the trivial statement $1 \otimes T^i = T^i$, since the trivial representation represents any element as the identity.
- The tensor product of the sign representation T^1 with itself produces the trivial representation. This is also natural since the sign representation represents any element as ± 1.
- $T^1 \otimes T^2 \cong T^2$. The tensor product of the sign representation with the 2-dimensional irrep gives the same 2-dimensional irrep (one isomorphic to it, actually).
- $T^3 \otimes T^3$ gives a 4-dimensional, reducible representation.

Section b)

Recall from the previous exercise that the matrix corresponding to a tensor product of linear maps is just what we would usually call the tensor product of the corresponding matrices:

$$\left[T^3(g) \otimes T^3(g)\right]_{bb',aa'} = \left[T^3(g)\right]_{ba} \left[T^3(g)\right]_{b'a'} \ .$$

We want to represent the tensor product as a conventional matrix, so we have to convert double indices to simple ones. Call $B = (bb')$ and $A = (aa')$, since a, a', b, b' take values 1 and 2 (T^3 is a 2-dimensional representation), we can think that A, B take values from 1 to 4 as $1 = (1,1)$, $2 = (1,2)$, $3 = (2,1)$, $4 = (2,2)$. Convince yourself that this corresponds to computing the tensor product in the usual way:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \otimes \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} p\begin{pmatrix} P & Q \\ R & S \end{pmatrix} & q\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \\ r\begin{pmatrix} P & Q \\ R & S \end{pmatrix} & s\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \end{pmatrix} = \begin{pmatrix} pP & pQ & qP & qQ \\ pR & pS & qR & qS \\ rP & rQ & sP & sQ \\ rR & rS & sR & sS \end{pmatrix}$$

Then the representation is ($\to$ means here representative in the $T^3 \otimes T^3$ representation to avoid writing the tensor product each time):

$$e \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1\,2) \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(1\,3) \to \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ -\sqrt{3} & -1 & 3 & \sqrt{3} \\ -\sqrt{3} & 3 & -1 & \sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix} \qquad (2\,3) \to \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ \sqrt{3} & -1 & 3 & -\sqrt{3} \\ \sqrt{3} & 3 & -1 & -\sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix}$$

$$(1\,2\,3) \to \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 3 \\ -\sqrt{3} & 1 & -3 & \sqrt{3} \\ -\sqrt{3} & -3 & 1 & \sqrt{3} \\ 3 & -\sqrt{3} & -\sqrt{3} & 1 \end{pmatrix} \qquad (1\,3\,2) \to \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} & -\sqrt{3} & 3 \\ \sqrt{3} & 1 & -3 & -\sqrt{3} \\ \sqrt{3} & -3 & 1 & -\sqrt{3} \\ 3 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix}$$

Section c)

The first thig we must know is the decomposition of $T^3 \otimes T^3 \equiv T^4$ in irreps. This is done by means of the product with the basis of characters:

$$(\chi_1|\chi_4) = 1 \ , \qquad (\chi_2|\chi_4) = 1 \ , \qquad (\chi_3|\chi_4) = 1 \ .$$

Thus, we conclude $T^4 = T^3 \otimes T^3 = T^0 \oplus T^1 \oplus T^2$. Knowing the irreps, we can try to do a brute force decomposition now looking for the subspaces of V_4 (the 4-dimensional representation space of $T^3 \otimes T^3$) in which the previous matrices act as the ones of the corresponding irrep. This is fairly easy to do for the trivial representation, but the lengthy computations involved will certainly discourage you from doing it in general. Let us just illustrate how it would be done. Let M be any of the previous 6 matrices, then we look for vectors v such that $Mv = v$ for all M (this is in the end what the trivial representation does: act as the identity). The representative for $(1\,2)$ forces $v^2 = v^3 = 0$, where we are denoting the components of v as $v = (v^1, v^2, v^3, v^4)$. You can check then that from any of the non-diagonal matrices you get the condition $v^1 = v^4$. The subspace in which the trivial representation T^0 acts is then:

$$V_0 = \{(v, 0, 0, v) \;/\; v \in \mathbb{C}\} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{\sqrt{2}}(1, 0, 0, 1) \right\} \; .$$

The systematic way to obtain the invariant subspaces in which each irrep acts involves using the projectors presented in the final sections of Chapter 3 in the lecture notes. For any irrep T^i, the invariant subspace in which it acts within the representation T is obtained by projecting with:

$$\pi^i = \frac{\dim(T^i)}{|S_3|} \sum_{g \in S_3} \chi_i^\star(g) T(g) \; .$$

Using the matrices from the previous section corresponding to $T^4 = T^3 \otimes T^3$ and the characters of the irreps, you should get:

$$\pi^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \; , \quad \pi^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; , \quad \pi^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \; .$$

Clearly, π^0 projects into the V_0 previously obtained. Similarly, π^1 projects into:

$$V_1 = \{(0, v, -v, 0) \;/\; v \in \mathbb{C}\} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{\sqrt{2}}(0, 1, -1, 0) \right\} \; .$$

Finally, π^2 has rank 2, and therefore projects into the 2-dimensional subspace:

$$V_2 = \{(x, y, y, -x) \;/\; x, y \in \mathbb{C}\} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{\sqrt{2}}(-1, 0, 0, 1), \frac{1}{\sqrt{2}}(0, 1, 1, 0) \right\} \; .$$

From these forms of the subspaces it is clear they are mutually orthogonal, as they should. Notice also that, since V_2 is 2-dimensional, we have to choose a basis to write the matrix representatives of the group elements, and different bases produce different (but equivalent) forms of the representation. The basis suggested in the previous expression is the one that reproduces the representatives given in the questions.